

PHYC56H3 Problem Set 1

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PROBLEM 1.1

(a) Let $x, y \in V$. I know that the vector addition and scalar multiplication operators in this vector space are defined as $x \oplus y = x + y + 7$ and $r \odot x = rx + 7(r - 1)$ respectively. Since addition and scalar multiplication give real numbers, the set is closed under addition and scalar multiplication trivially. The work of proving/disproving the axioms is shown below. This set is not a vector space because the axiom of multiplication compatibility addition does not hold. The distributivity of vector addition and scalar multiplication also do not hold.

Commutativity of Addition

$$x \oplus y = x + y + 7 = y + x + 7 = y \oplus x$$

Associativity of Addition

$$\begin{aligned}(x \oplus y) \oplus z &= (x + y + 7) + z + 7 \\ &= x + (y + z + 7) + 7 \\ &= x \oplus (y \oplus z)\end{aligned}$$

$\exists$ Identity Element of Addition

$$x \oplus (-7) = x + (-7) + 7 = x$$

$\exists$ Inverse Element of Addition

$$x \oplus (-x - 7) = x + (-x - 7) + 7 = 0$$

Preservation of Scale

$$1 \odot x = 1x + 7 - 7 = x$$

Multiplication Compatibility

$$\begin{aligned}r \odot (w \odot x) &= r \odot (wx + 7w - 7) \\ &= r(wx + 7w - 7) + 7r - 7 = rwx + 7rw - 7r + 7r - 7 \\ &= rwx + 7rw - 7 = (rw) \odot x\end{aligned}$$

Distributivity of Vector Addition

Let w be another scalar in our field.

$$\begin{aligned}(r + w) \odot x &= (r + w)x + 7(r + w) - 7 \\ &= rx + wx + 7r + 7w - 7 = (rx + 7r - 7) + (wx + 7w) \\ &= r \odot x + (wx + 7w) \neq r \odot x + w \odot x\end{aligned}$$

Distributivity of Scalar Multiplication

$$\begin{aligned}r \odot (x \oplus y) &= r(x \oplus y) + 7r - 7 \\ &= r(x + y + 7) + 7r - 7 = rx + ry + 7r + 7r - 7 \\ &= (rx + 7r - 7) + (ry + 7r) = r \odot x + (ry + 7r) \neq rx \odot ry\end{aligned}$$

(b) I already found the identity element for scalar multiplication from our preservation of scale proof. It turns out to be $r = 1, r \in \mathbb{R}$.

(c) Our identity element of addition is $(-7) \in V$. I also found that the inverse element of addition is $-x = (-x-7) \in V$ so that $x + (-x) = 0$.

PROBLEM 1.2

(a) I recall the general inner product for complex vectors with n components. $\langle \mathbf{v} | \mathbf{v} \rangle = \overline{v_1}v_1 + \overline{v_2}v_2 + \dots + \overline{v_n}v_n$. Firstly, for $|a\rangle$, there is no need to normalise $|a\rangle$ as shown below.

$$\langle a|a\rangle = \overline{\left(\frac{1}{\sqrt{2}}\right)}\frac{1}{\sqrt{2}} + \overline{\left(\frac{i}{2}\right)}\frac{i}{2} + \overline{\left(\frac{1}{2}\right)}\frac{1}{2} = 1$$

$\sqrt{1} = 1$, so $|a\rangle$ is normalised, and our normalised vector is $|a\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{i}{2}|2\rangle + \frac{1}{2}|3\rangle$. I do the same for $|b\rangle$, but it turns out that it is, unlike $|a\rangle$, not normalised.

$$\langle b|b\rangle = \overline{\left(\frac{1}{\sqrt{3}}\right)}\frac{1}{\sqrt{3}} + \overline{\left(\frac{i}{\sqrt{3}}\right)}\frac{i}{\sqrt{3}} = \frac{2}{3}$$

$$\sqrt{\frac{2}{3}} \neq 1, \text{ so it is not normalised.}$$

In order to normalise it, I simply took the reciprocal of the normal and multiplied it with $|b\rangle$. So our normalised vector is:

$$|b\rangle = \sqrt{\frac{3}{2}} \left(\frac{1}{\sqrt{3}}|1\rangle + \frac{i}{\sqrt{3}}|2\rangle \right)$$

(b) In order to check orthogonality, we take the inner product of $|a\rangle$ and $|b\rangle$. It turns out that, in this case, the order of the inner product does not matter.

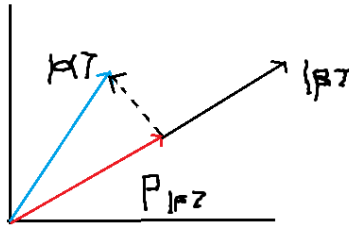
$$\begin{aligned} \langle a|b\rangle &= \overline{\left(\frac{1}{\sqrt{2}}\right)}\frac{1}{\sqrt{3}} + \overline{\left(\frac{i}{2}\right)}\frac{i}{\sqrt{3}} + \overline{\left(\frac{1}{2}\right)}0 \\ &= \frac{1}{\sqrt{6}} + \frac{1}{2\sqrt{3}} = \frac{1}{6}(\sqrt{6} + \sqrt{3}) \\ &= \langle b|a\rangle \neq 0 \end{aligned}$$

I find that the vectors $|a\rangle$ and $|b\rangle$ are not orthogonal to each other. As a further step, I find a vector that is orthogonal to both $|a\rangle$ and $|b\rangle$. I do this with a cross product.

$$|a\rangle \times |b\rangle = \begin{vmatrix} |1\rangle & |2\rangle & |3\rangle \\ \frac{1}{\sqrt{2}} & \frac{i}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{3}} & \frac{i}{\sqrt{3}} & 0 \end{vmatrix} = |1\rangle \left(\frac{1}{2\sqrt{3}} \right) + |2\rangle \left(\frac{1}{2\sqrt{3}} \right) + |3\rangle \left(\frac{i}{\sqrt{6}} - \frac{i}{2\sqrt{3}} \right)$$

PROBLEM 1.3

(a)



Sketch of the effect of $P_{|\beta\rangle}$ on $|\alpha\rangle$ in $\mathbb{R}^2$.

(b) I wish to calculate $P_{|\beta\rangle}^2 |\alpha\rangle = P_{|\alpha\rangle} (P_{|\beta\rangle} |\alpha\rangle)$.

$$\langle P_{|\beta\rangle} |\alpha\rangle, P_{|\beta\rangle} |\alpha\rangle \rangle =$$

(c) The example gives us a orthonormal basis $\{|1\rangle, |2\rangle, |3\rangle\}$, so the projections are much easier to calculate since the orthonormal basis vectors are unit vectors.

(d)

PROBLEM 1.4

I start off by taking the square root of both sides. I also note the normals are defined as $\sqrt{\langle\alpha|\alpha\rangle} = \|\alpha\|$ and $\sqrt{\langle\beta|\beta\rangle} = \|\beta\|$ similarly.

$$\begin{aligned} \langle\alpha|\beta\rangle &= \sqrt{\langle\alpha|\alpha\rangle} \sqrt{\langle\beta|\beta\rangle} \\ &= \|\alpha\| \|\beta\| \end{aligned}$$

Using Gram-Schmidt Orthonormalization, or the hint given in the problem set, we can introduce a new, orthogonal, vector $|\gamma\rangle$. Where $|\gamma\rangle = |\beta\rangle - \frac{\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle} |\alpha\rangle$. According to Gram-Schmidt, $|\gamma\rangle$ is the projection of $|\beta\rangle$ onto the plane orthogonal to $|\alpha\rangle$. A sketch of such a projection is done in an earlier question. I verify that $|\gamma\rangle$ and $|\alpha\rangle$ are orthogonal by taking the inner product .

$$\begin{aligned} \langle\gamma|\alpha\rangle &= \langle|\beta\rangle - \frac{\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle} |\alpha\rangle, |\alpha\rangle\rangle \\ &= \langle\beta|\alpha\rangle - \frac{\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle} \langle\alpha|\alpha\rangle \text{ by linearity} \\ &= \langle\beta|\alpha\rangle - \langle\alpha|\beta\rangle = 0 \end{aligned}$$

Rearranging $|\gamma\rangle$ for $|\beta\rangle$, and then squaring the normal of both sides, I get

$$\begin{aligned}
|\beta\rangle &= |\gamma\rangle + \frac{\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle} \\
\Leftrightarrow ||\beta\rangle|^2 &= ||\gamma\rangle|^2 + \left| \frac{\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle} \right|^2 ||\alpha\rangle|^2 \\
&= ||\gamma\rangle|^2 + \frac{|\langle\alpha||\beta\rangle\rangle|^2}{||\alpha\rangle|^2} \\
&\geq \frac{|\langle\alpha||\beta\rangle\rangle|^2}{||\alpha\rangle|^2} \text{ Normal of } |\gamma\rangle \geq 0 \text{ because of positive-definiteness.}
\end{aligned}$$